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Certain Questions of Consistency in S-Matrix Theory*

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Certain questions of consistency in S-matrix theory arising from unitarity, analyticity, and crossing symmetry requirements are examined in a few special instances for two-body scattering amplitudes. It is proved that at least in these cases, the formalism does not lead to any contradiction.

I. INTRODUCTION

IN his investigation of the constraints imposed by analyticity and unitarity on two-body scattering amplitudes, Martin has remarked on a certain consistency problem in S-matrix theory.[1] Thus, in a scattering process for which there is an elastic interval in two of the channels, if the double spectral functions are given in one of these intervals, the scattering amplitude gets determined uniquely in a way that does not appear to guarantee its crossing symmetry when such a symmetry is expected on general grounds. A similar situation arises in the author's proof[2] that in a two-particle scattering process, if all but a finite number of partial waves are given, the remaining partial waves are uniquely determined (possibly up to an additive S-wave constant) if the scattering amplitude either has crossing symmetry or satisfies elastic unitarity in one of the crossed channels. When the amplitude meets with both these requirements, consistency would therefore demand that these two determinations should in fact be the same. Since this question is of some importance in S-matrix theory, the compatibility of the assumptions of analyticity, unitarity, and crossing symmetry are verified in a few special instances in this paper. The discussion is divided into two parts. In Sec. II, the proofs of two uniqueness theorems due to Martin[1] are summarized, emphasizing those aspects relevant for our discussion. Section III contains the main body of our results and is based in part on Sec. II.

II. MARTIN'S THEOREMS

We consider the elastic scattering of two nonidentical spinless particles of equal mass m, and denote the Mandelstam variables for the process by s, t, and u. These are subject to the usual constraint

$$s+t+u=4m^2. \tag{1}$$

The double spectral function which is nonvanishing in part of the region $s\geqslant 4m^2$, $t\geqslant 4m^2$ in the s–t plane is denoted by $\rho_{st}(s,t)$, and those nonvanishing in corresponding regions in the t–u and u–s planes are denoted by $\rho_{tu}(t,u)$ and $\rho_{us}(u,s)$.

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[1] A. Martin, Phys. Rev. Letters **9**, 410 (1962).

[2] A. P. Balachandran, Phys. Rev. **132**, 894 (1963).

With these preliminaries, we may now turn to the statement and brief proofs of Martin's results.

(1) If there are intervals $4m^2\leqslant s<s_1$, $4m^2\leqslant t<t_1$ in the s and t channels where scattering is purely elastic, and if the double spectral functions $\rho_{st}(s,t)$ and $\rho_{su}(s,u)$ are given in the strip $4m^2\leqslant s<s_1$, the scattering amplitude is uniquely determined.

Let $F(s,t,u)$ and $F'(s,t,u)$ denote two possible scattering amplitudes consistent with the hypotheses. Since the double spectral functions of F and F' coincide for $4m^2\leqslant s<s_1$, where elastic unitarity may be used, it is true that[1]

$$F(s,t,u)-F'(s,t,u)=\sum_{n=0}^{N} a_n(s)t^n, \tag{2}$$

where N denotes the number of subtractions in the Mandelstam representation. The right-hand side of Eq. (2) is regular for every finite t and fixed s. The absorptive part of $F-F'$ in the t channel therefore vanishes. If $G_l(t)$ and $G_l'(t)$ denote the partial-wave amplitudes in the t channel, it follows that

$$\mathrm{Im}G_l(t)=\mathrm{Im}G_l'(t),\quad \text{for}\quad t\geqslant 4m^2. \tag{3}$$

For $4m^2\leqslant t<t_1$, using elastic unitarity, we deduce from Eq. (3) that

$$\mathrm{Re}G_l(t)=\pm\mathrm{Re}G_l'(t). \tag{4}$$

Martin's argument[1] then shows that only the solution $\mathrm{Re}G_l(t)=+\mathrm{Re}G_l'(t)$ is acceptable for $l>N$. Thus, $G_l(t)=G_l'(t)$ for $l>N$ and $4m^2\leqslant t<t_1$, and, by analytic continuation, for all t. $F-F'$ may thus be written as

$$F(s,t,u)-F'(s,t,u)=\sum_{n=0}^{N} b_n(t)s^n. \tag{5}$$

Comparing (2) and (5), we conclude that $a_n(s)$ and $b_n(s)$ are polynomials of degree N in s:

$$a_n(s)=\sum_{r=0}^{N} a_{n,r}s^r,\qquad b_n(s)=\sum_{r=0}^{N} b_{n,r}s^r. \tag{6}$$

Since, however, partial waves are bounded at infinity, and $a_n(s)$ and $b_n(s)$ are linear combinations of partial waves with coefficients which remain bounded for large

s, only $a_{n,0}$ and $b_{n,0}$ may be nonzero constants. Finally, (2) and (5) reveal that $a_0(s)=b_0(s)=a_{0,0}=b_{0,0}$ while all the other a_n and b_n must be zero. Thus,

$$F(s,t,u)-F'(s,t,u)=a_{0,0}. \tag{7}$$

But, if $a_{0,0}$ were not zero, there would be no inelastic S-wave scattering in s and t channels.[1] Since this is not acceptable, $a_{0,0}=0$ and

$$F(s,t,u)=F'(s,t,u). \tag{8}$$

(2) If there is an energy interval $4m^2\leqslant s<s_1$ in the s channel where scattering is purely elastic and if the double spectral functions $\rho_{st}(s,t)$ and $\rho_{su}(s,u)$ are given in the corresponding strip, the scattering amplitude is uniquely determined provided it satisfies crossing symmetry in the s and t (or s and u) variables, that is, if $F(s,t,u)=F(t,s,u)$ [or $F(s,t,u)=F(u,t,s)$].

The simplest way of proving this assertion is to observe that crossing symmetry in s and t implies that there is an elastic interval $4m^2\leqslant t<s_1$ in the t channel, which then reduces this case to the previous one. Alternatively, one may appeal to a pattern of proof presented in Ref. 2. The latter is the one relevant for our discussion, since, *a priori*, the former need not even lead to a crossing symmetric amplitude.

III. ON THE QUESTIONS OF CONSISTENCY

It will prove convenient to divide this section into four parts. The first two of these attempt to resolve the problem posed by Martin,[1] while the remaining two refer to a situation discussed in Ref. 2. It is important to observe that in what follows, we do not attempt to prove the existence of various solutions we encounter, but merely assume that such functions can in fact be constructed. Note also that our methods of proof are quite similar to Martin's.[1]

(1) Here we assume that there exists an interval $4m^2\leqslant s<s_1$ in the s channel where scattering is purely elastic. The double spectral functions $\rho_{st}(s,t)$ and $\rho_{su}(s,u)$ are given in this strip. If, now, we assume crossing symmetry in s and t, we obtain one determination of the scattering amplitude which we denote by $F(s,t,u)$. On the other hand, if we assume that there exists an interval $4m^2\leqslant t<t_1$ in the t channel where scattering is elastic, we obtain another determination of the scattering amplitude which we denote by $F'(s,t,u)$. It is to be proved that $F(s,t,u)=F'(s,t,u)$.

Since the double spectral functions of F and F' coincide for $4m^2\leqslant s<s_1$, where elastic unitarity can be applied, we find[1]

$$F(s,t,u)-F'(s,t,u)=\sum_{n=0}^{N}a_n(s)t^n. \tag{9}$$

Crossing symmetry of F implies

$$F'(s,t,u)-F'(t,s,u)=\sum_{n=0}^{N}a_n(t)s^n-\sum_{n=0}^{N}a_n(s)t^n. \tag{10}$$

For the absorptive parts in the t variable, Eq. (10) gives

$$A_t'(s,t,u)-A_s'(t,s,u)=\sum_{n=0}^{N}\mathrm{Im}a_n(t)s^n. \tag{11}$$

If we denote the partial-wave amplitudes of $F'(s,t,u)$ and $F'(t,s,u)$ in the t channel by $G_l'(t)$ and $G_l''(t)$ respectively, it follows that

$$\mathrm{Im}G_l'(t)=\mathrm{Im}G_l''(t)\quad\text{for}\quad l>N\quad\text{and}\quad t\geqslant 4m^2. \tag{12}$$

But, by hypothesis, both $G_l'(t)$ and $G_l''(t)$ satisfy elastic unitarity for $4m^2\leqslant t<t_1$, and therefore[1]

$$G_l'(t)=G_l''(t)\quad\text{for}\quad l>N\quad\text{and all } t. \tag{13}$$

Thus,

$$F'(s,t,u)-F'(t,s,u)=\sum_{n=0}^{N}b_n(t)s^n. \tag{14}$$

Comparing Eqs. (10) and (14), we conclude that $a_n(s)$ is a polynomial of degree N in s. The argument following Eq. (6) now applies, and finally,

$$F(s,t,u)=F'(s,t,u), \tag{15}$$

which proves the result.

(2) The double spectral functions are given in the interval $4m^2\leqslant s<s_1$ where scattering is purely elastic. If we assume crossing symmetry between s and u channels, we are provided with one determination of the scattering amplitude which may be denoted by $F(s,t,u)$. If we assume elastic unitarity in some interval $4m^2\leqslant t<t_1$ in the t channel, we obtain another determination of the scattering amplitude which may be denoted by $F'(s,t,u)$. It is to be proved that $F(s,t,u)=F'(s,t,u)$.

Since the double spectral functions of F and F' coincide for $4m^2\leqslant s<s_1$, we find[1]

$$F(s,t,u)-F'(s,t,u)=\sum_{n=0}^{N}a_n(s)t^n. \tag{16}$$

The crossing symmetry of F implies that

$$F'(s,t,u)-F'(u,t,s)=\sum_{n=0}^{N}a_n(u)t^n-\sum_{n=0}^{N}a_n(s)t^n. \tag{17}$$

It should be observed that in Eq. (16), the functions $a_n(s)$ may have no left-hand cut, for if they did, (16) would violate fixed t-dispersion relation. Thus, $a_n(s)$ may have poles in the interval $0\leqslant s<4m^2$ and a cut running from $4m^2$ to ∞. If we fix u, the singularities of $a_n(s)$ are therefore contained along the line $\mathrm{Re}t\leqslant 4m^2-u$. Thus, the absorptive part of (17) in the t channel vanishes for $u>0$ and by analytic continuation, for all u. If we denote the partial-wave amplitude of $F'(s,t,u)$ in the t channel by $G_l'(t)$ and that of $F'(u,t,s)$ by $G_l''(t)$, it follows that

$$\mathrm{Im}G_l'(t)=\mathrm{Im}G_l''(t)\quad\text{for}\quad t\geqslant 4m^2. \tag{18}$$

But F' satisfies elastic unitarity in some interval $4m^2 \leqslant t < t_1$ in the t channel, and so

$$\mathrm{Re}G_l'(t) = \mathrm{Re}G_l''(t) \quad \text{for} \quad 4m^2 \leqslant t < t_1 \quad \text{and} \quad l > N. \tag{19}$$

By analytic continuation, it follows that

$$G_l'(t) = G_l''(t) \tag{20}$$

for all t and $l > N$. This gives

$$F'(s,t,u) - F'(u,t,s) = \sum_{n=0}^{N} b_n(t) s^n, \tag{21}$$

where $b_n(t)$ may have poles in the interval $0 \leqslant t < 4m^2$. Comparing (17) and (21), we find

$$\sum_{n=0}^{N} a_n(u) t^n - \sum_{n=0}^{N} a_n(s) t^n = \sum_{n=0}^{N} b_n(t) s^n. \tag{22}$$

If we fix t at some $t > 4m^2$, $a_n(u)$ will have singularities in the s plane along the line $\mathrm{Re}s \leqslant 4m^2 - t < 0$. We therefore take the discontinuity of (22) in s for fixed $t > 4m^2$ to obtain

$$\mathrm{Im}a_n(s) = 0 \quad \text{for} \quad s \geqslant 0. \tag{23}$$

$a_n(s)$ is thus a polynomial of degree N in s. Since $a_n(s)$ is a linear combination of partial waves with coefficients bounded at infinity, the argument following Eq. (6) applies. Finally,

$$F(s,t,u) = F'(s,t,u) \tag{24}$$

which completes the proof.

(3) We are given all partial waves for $l > M$ in the s channel, where M is some integer. If we assume crossing symmetry between s and t channels, we have one determination of the scattering amplitude[2] which we denote by $F(s,t,u)$. If we assume elastic unitarity for $4m^2 \leqslant t < t_1$ in the t channel, we have another determination of the scattering amplitude, which we denote by $F'(s,t,u)$.[2] It is to be proved that $F(s,t,u) = F'(s,t,u)$.

By hypothesis,

$$F(s,t,u) - F'(s,t,u) = \sum_{n=0}^{M} a_n(s) t^n. \tag{25}$$

Crossing symmetry of F implies that

$$F'(s,t,u) - F'(t,s,u) = \sum_{n=0}^{M} a_n(t) s^n - \sum_{n=0}^{M} a_n(s) t^n. \tag{26}$$

The discontinuity of (26) in t for $t \geqslant 4m^2$ is a polynomial in s:

$$A_t'(s,t,u) - A_s'(t,s,u) = \sum_{n=0}^{M} \mathrm{Im}a_n(t) s^n. \tag{27}$$

If $G_l'(t)$ and $G_l''(t)$ denote the partial-wave amplitudes of $F'(s,t,u)$ and $F'(t,s,u)$ in the t channel, (27) gives

$$\mathrm{Im}G_l'(t) = \mathrm{Im}G_l''(t) \quad \text{for} \quad l > M \quad \text{and} \quad t \geqslant 4m^2. \tag{28}$$

By hypothesis, $F'(s,t,u)$ satisfies elastic unitarity in the t variable for $4m^2 \leqslant t < t_1$. Further, the higher partial waves that are initially given must satisfy elastic unitarity in the same energy interval, for if they did not, it is clear that there is an inconsistency in the assumptions underlying the problem. Thus, both $G_l'(t)$ and $G_l''(t)$ satisfy elastic unitarity for some interval in t. Therefore, (28) gives $\mathrm{Re}G_l'(t) = \pm \mathrm{Re}G_l''(t)$ for $l > M$ and t in this interval. This means that $G_l'(t) = G_l''(t)$ for $l > \max\{M,N\}$ and all t and

$$F'(s,t,u) - F'(t,s,u) = \sum_{n=0}^{\max\{M,N\}} b_n(t) s^n. \tag{29}$$

Comparing (26) and (29), we conclude that $a_n(s)$ is a polynomial in s. The proof can now be completed by following a previous discussion.

(4) Here we modify the situation in (3) by assuming that $F(s,t,u)$ is crossing symmetric in s and u instead of in s and t. Equation (26) is replaced by

$$F'(s,t,u) - F'(u,t,s) = \sum_{n=0}^{M} a_n(u) t^n - \sum_{n=0}^{M} a_n(s) t^n. \tag{30}$$

This is similar to Eq. (17). We therefore conclude that

$$F(s,t,u) - F'(s,t,u) = a_{0,0}, \tag{31}$$

where $a_{0,0}$ is a constant. But (31) shows that the spectra of $F(s,t,u)$ and $F'(s,t,u)$ in the t channel are identical. Therefore, since $F'(s,t,u)$ satisfies elastic unitarity for $4m^2 \leqslant t < t_1$ in the t channel, $F(s,t,u)$ also satisfies it in this interval. The discussion in Sec. II then assures us that $a_{0,0} = 0$ and $F(s,t,u) = F'(s,t,u)$.

In conclusion, we wish to emphasize that in the foregoing discussion, we have merely verified that the S-matrix formalism does not lead to inconsistencies in a few special instances. This, therefore, does not constitute a proof of its consistency even for two-body scattering amplitudes.

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